

Stability and breakdown of integrability in quantum many-body dynamics

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In quantum many-body theory, no generic microscopic principle at the origin of complex dynamics is known. In this Letter, we present a method for solving the dynamics of a rather general class of quantum integrable and nearly integrable systems, which leads to a natural distinction between regular and irregular behaviors. The basic idea is, that the dynamics of the integrable quantum system is described by some underlying classical one, which can be analyzed using the powerful tools of classical theory of motion. We apply the approach to the Dicke Hamiltonian subjected to a periodically driven detuning. This represents generic example of a dynamically perturbed integrable model. We show that scattering in the classical phase space can drive the quantum model close to thermal equilibrium. Interestingly, this happens in the fully quantum regime, where physical observables do not show any dynamic chaotic behavior.

Introduction – Many aspects of the transition from regular dynamics of an integrable system to erratic behavior of a complex system are understood in classical mechanics. On the one hand, there is the Kolmogorov-Arnold-Moser (KAM) theorem [1–3], that proves the stability of weakly perturbed integrable systems. On the other hand, a variety of mechanisms leading to chaos and eventually to the ergodic exploration of phase space have been found. For quantum systems, there exist semiclassical [4] and phenomenological (e.g. random matrices [4, 5]) descriptions of quantum chaotic phenomena, but there remains an important conceptual gap between regular and complex behaviors. In this Letter we approach non-trivial quantum systems away from the semiclassical limit and gain microscopic insight into the emergence of irregularity by developing a theory of *dynamical* perturbation from integrability.

A good starting point to approach regular dynamics of non-trivial quantum systems are Bethe ansatz (BA) integrable models, which possess a complete set of integrals of motion. The exact solutions of time-independent BA many-body solvable systems played a crucial role in the understanding of various fundamental phenomena and concepts in physics. Famous examples are the solutions for the Ising model, the Heisenberg spin chain, the one-dimensional Hubbard model or the Lieb-Liniger gas [6]. However, the non-equilibrium dynamics of these models are rich [7, 8] and much more difficult to be calculated within the BA than the static properties. Formulating a dynamical deviation from integrability is therefore not only a conceptual, but also a technical challenge.

The main finding presented in this Letter is that there exists an exact description of the quantum integrable model in terms of a classical many-body interacting system. Dynamical deviation from integrability for the quantum system can then be understood in terms of the classical system, for which powerful tools such as the KAM theorem are available. We will first derive these underlying classical systems for Gaudin models and then, as an example, analyze their behavior in a periodically driven Dicke model.

Dynamical ansatz and separation of variables – Within the algebraic BA [9], the transfer matrix generates integrals of motion, which will be sensitive to the breaking of integrability. All of them depend on rapidity parameters, λ_m ($m = 1, \dots, M$, where M is the number of excitations or particles in the system), which, in the time-independent case, are determined by the Bethe equations. Our approach is to introduce time-dependent rapidities, which describe the dynamics of a certain class of BA integrable models with arbitrary time-dependence of parameters. These rapidities are equivalent to position variables of a classical many-body system. The first step in the derivation of the system of classical equations of motions is a *separation of variables* for the quantum mechanical wave function [3].

To be specific, we restrict the following discussion to the broad class of so-called Gaudin models [6, 12], but the separation of variables can be applied to any integrable model, such as e.g. quantum spin chains [5]. Gaudin models are relevant to a number of physical systems. For example, Dicke models describe cavity QED, Richardson models have been applied to mesoscopic superconductivity and central spin models to quantum dots and NV centers in diamonds. Gaudin-type models represent a generic situation in the following sense: BA-solvable models may depend on a number of parameters, one of them controls the so-called “quasiclassical” behavior. One can expand a generic integrable model in a Taylor series of this parameter. The first nontrivial term of this expansion is a generic Gaudin-type model. Therefore, the quantum phenomenon we observe here for a small but finite value of this parameter can be expected to persist in general case.

For a Gaudin model with N sites, resp. inhomogeneity parameters, Sklyanin [3] introduced separated variables v_j ($j = 1, \dots, N$) as operator zeros of an off-diagonal element of the L -matrix (see Supplementary Material). For a wavefunction in a representation of u_j , one can then introduce momentum variables $v_j = (d/du_j) - \Lambda(u_j)$, where the concrete form of the “gauge potential” $\Lambda(u_j)$ depends on the symmetry and the concrete realization of

the Gaudin model. Generically, it has a singular structure, $\Lambda(u) \sim \sum_{j=0}^N l_j(u - z_j)^{-1}$ with some given local spin quantum numbers l_j , provided u is sufficiently close to the vortices (or monopoles) situated at points z_j in the complex plain. This structure is related to effective 2D Coulomb gas, similar to the situation in the Hall effect. In addition to these separated variables, there exists as well a special overall symmetry operator u_0 .

The main advantage of using separated variables is that the transfer T -operator of the Gaudin algebra, which represents a generating function for the Hamiltonian and the conserved quantities, is written as a combination of separated kinetic terms, $T(u_j) = v_j^2$. Therefore, the Hamiltonian has the form of particles moving in a monopole gauge potential given by $\Lambda(u)$ and the time-dependent Schrödinger equation for the wave function $\Psi(\{u_j\})$ can be separated,

$$i\dot{\psi}(u) = \frac{d^2\psi(u)}{du^2} - 2\Lambda(u)\frac{d\psi(u)}{du} + (\Lambda^2(u) - \Lambda'(u))\psi(u) \quad (1)$$

for every $u_j \equiv u$, with $\Psi(\{u_j\}) = \prod_{j=0}^N \psi(u_j)$.

Classical system – Our next conceptual step is to show that each differential equation (1) for one individual separated variable $u = u_j$ is equivalent to a Hamiltonian system of auxiliary classical particles. It is straightforward to represent the factors of the wave function $\Psi(\{u_j\})$ in a product form,

$$\psi(u, t) = \mathcal{C}(t)\mathcal{N}(u) \prod_{m=1}^M (u - \lambda_m(t)), \quad (2)$$

where the factors $\mathcal{C}(t)$ and $\mathcal{N}(u)$ ensure normalization of the wave function. The moving roots $\lambda_m(t)$ ($m = 1, \dots, M$) of the wave function take time-dependence into account. This method for treating time dependence is due to Calogero [6]. Substituting the product form ansatz (2) into the Schrödinger equation (1), a system of first-order differential equations can be obtained: $i\dot{\lambda}_m = \Phi_m(\{\lambda_n\})$ where $n, m = 1 \dots M$, and the function $\Phi_m(\{\lambda_n\})$ is entirely determined by the explicit form of the potential $\Lambda(\lambda)$, $\Phi_m(\{\lambda_n\}) = a(\lambda_m)\Lambda(\lambda_m) + \sum_{n=1}^M a(\lambda_m)(\lambda_m - \lambda_n)^{-1}$, where $a(\lambda_m) = \prod_{j=1}^N (\lambda_m - z_j)$. This system of differential equations are equations of motion for auxiliary classical particles which move according to the Lagrangian

$$L = \sum_{m=1}^M \dot{x}_m^2 - \sum_{m=1}^M \frac{\Phi_m^2(\{\lambda_n\})}{a(\lambda_m)}, \quad (3)$$

where $x_m = \int d\lambda_m a^{-1/2}(\lambda_m)$ and $a(\lambda_m) = \prod_{j=1}^N (\lambda_m - z_j)$. The corresponding Hamiltonian is therefore $H = \sum_m p_m^2 + V(\{\lambda_n\})$ with single-body, two-body and three-

body potentials:

$$\begin{aligned} \frac{1}{4}V(\{\lambda_n\}) &= \sum_{m=1}^M \frac{\Lambda^2(\lambda_m)}{a(\lambda_m)} - \sum_{\substack{n,m=1 \\ n \neq m}}^M \frac{\Lambda(\lambda_n)}{\lambda_m - \lambda_n} \\ &+ 2 \sum_{\substack{n,m=1 \\ n \neq m}}^M \frac{\Lambda(\lambda_n)}{(\lambda_m - \lambda_n)^2} + 2 \sum_{\substack{l,m,n=1 \\ n \neq m \neq l}}^M \frac{a(\lambda_m)}{(\lambda_m - \lambda_n)(\lambda_m - \lambda_l)}. \end{aligned} \quad (4)$$

This is the classical many-body integrable model which governs the motion of the poles of the wave function and motivates a new correspondence principle for characterizing quantum dynamics: the dynamics of a quantum system can be called regular or nearly integrable if the trajectories of the auxiliary particles determined by Eqs. (3,4) reside on KAM tori and the breakdown of these tori can be seen as the criterion for irregular quantum dynamics. In the framework of our approach two different cases of perturbing the integrable system can be identified: one (vertical) corresponds to perturbations which leave the structure of separated variables untouched, while the other (horizontal) mixes separated variables. The first case corresponds to dynamical driving of parameters of integrable systems while the second case can be achieved by adding new terms to the Hamiltonian. The most generic perturbation would be a mixture of both. In terms of the classical system, the horizontal perturbation can be described by attributing a "color" (additional index j) to the coordinates and momenta of the auxiliary particles. In what follows we are going to test our principle on the first class of perturbations.

Example: Dicke model – As an illustration of our formalism, we consider the Dicke model which represents a non-trivial physical example of the class of Gaudin models. This model, described by only one separated variable, is sufficiently generic to reveal the universal features of Gaudin models perturbed in the vertical direction. It was introduced by Dicke [14] and is widely used in the context of interaction of light and matter in quantum optics [15]. One particular application of the Dicke Hamiltonian is the description of the Bose-Einstein condensate in an optical cavity [16].

The model treats the interaction between a single-mode bosonic field (b and b^\dagger , photon annihilation and creation operators) and an ensemble of two-level systems (regarded as effective spins detuned from the bosonic modes by Δ) in the rotating wave approximation. The quantum Hamiltonian reads

$$H_D = \Delta S^z + g(b^\dagger S^- + b S^+), \quad (5)$$

where $S^a = \sum_{j=1}^{2S} \frac{\sigma_j^a}{2}$ is a collective spin operator. The total number of excitations $M = b^\dagger b + S^z + S$ is a conserved quantity. Therefore, the relative strength of the detuning Δ/g is the only free parameter in the system in a given sector with well defined quantum numbers M and S .

We carry out the explicit Bethe ansatz solution of this model for which $a(\lambda) = \lambda$ and $\Lambda(\lambda) = (\Delta - \lambda + 2S/\lambda)/2$ in the Supplementary Material, where we also show that the underlying classical many-body system governing the evolution of the spectral data in this model is a so-called Inozemtsev-Meshcheryakov model [8]. It describes the motion of M particles in a nonlinear external potential, which interact via a long-range interaction of Calogero-Moser type,

$$H_{IM} = \sum_{m=1}^M [p_m^2 + V(x_m)] + \sum_{\substack{n,m=1 \\ n \neq m}}^M \frac{1}{(x_n - x_m)^2} + \frac{1}{(x_n + x_m)^2}. \quad (6)$$

The potential takes the form $V(x_m) = \frac{1}{64}x_m^6 + \beta x_m^4 + \gamma x_m^2 + 16S^2x_m^{-2}$, where $\beta = -\frac{\Delta(t)}{8}$ and $\gamma = \frac{1}{4}(\Delta^2(t) - \dot{\Delta}(t) - 4S + M - 1)$. In spite of the nonlinear potential, this classical model is integrable.

Here we brake the integrability by time-dependent driving of the detuning. The model still pertains its form as in Eq. (6), however its parameters now depend on time. Concretely, we consider the following setup: At $t = 0$ the system is prepared in its ground state at $\Delta = \Delta_0$. Then we evaluate numerically its time evolution under the periodic detuning $\Delta(t) = \Delta_0 \cos(\omega t)$. We solve the dynamical equations by using a Runge-Kutta integration scheme. The rapidities can be obtained from the coefficients of the wave functions by finding the roots of symmetric polynomials. For these illustrative purposes, we choose a small number of excitations, $M = 4$, $S = 6$ and a strong amplitude of the detuning $\Delta_0/g = 5$, such that the bosonic modes are highly occupied initially, $N_b = \langle b^\dagger b \rangle \approx 3.2$, and the population of excited spins is small. The high driving amplitude causes strong dynamical redistribution of excitations between bosonic and spin degrees of freedom. If the driving frequency is sufficiently slow and non-resonant, dynamics remain almost adiabatic and observables are expected to exhibit periodic oscillations along the instantaneous ground state values.

In Fig. 1(a) such an example is shown: At frequency $\omega = 3.57g/\hbar$, there are regular oscillations of the boson populations $N_b(t)$ between 3.2 and 0.2. The rapidities, which correspond to the position variables of the classical model (6), are monitored stroboscopically after each cycle (i.e. at time $t_p = 2\pi p/\omega$, $p = 0, \dots, P$, where $P = 4000$ in the present case) by collapsing them onto a single complex plane. Fig. 1(c) shows that in this non-resonant case the rapidities cluster on circles located around the ground state positions. These circles indicate the existence of stable KAM-tori in the 16-dimensional phase space of the classical system and according to our principle of correspondence between dynamics of the quantum and the auxiliary classical system, we can classify such behavior as nearly integrable.

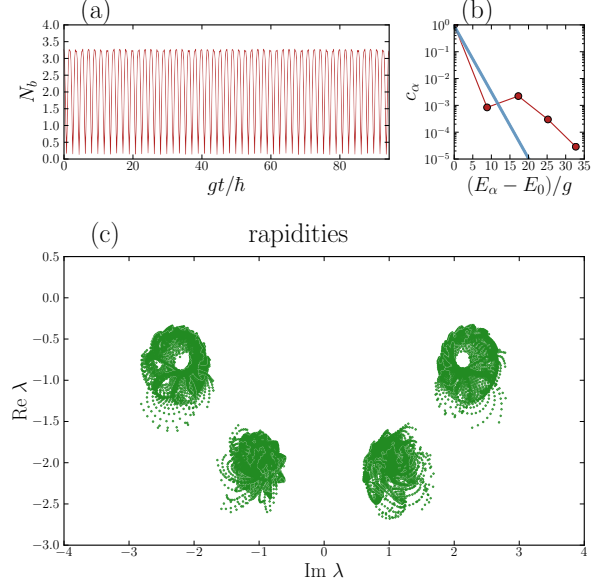


FIG. 1: Dynamics of the Dicke model driven non-resonantly with amplitude $\Delta_0/g = 5$ and a frequency $\omega = 3.57g/\hbar$, $S = 6$ and $M = 4$. (a) The boson occupation number N_b monitored over some interval of time, (b) the weights of eigenstates (7) c_α and (c) the stroboscopic maps of all rapidities λ_m , $m = 1, \dots, M$ after 4000 cycles.

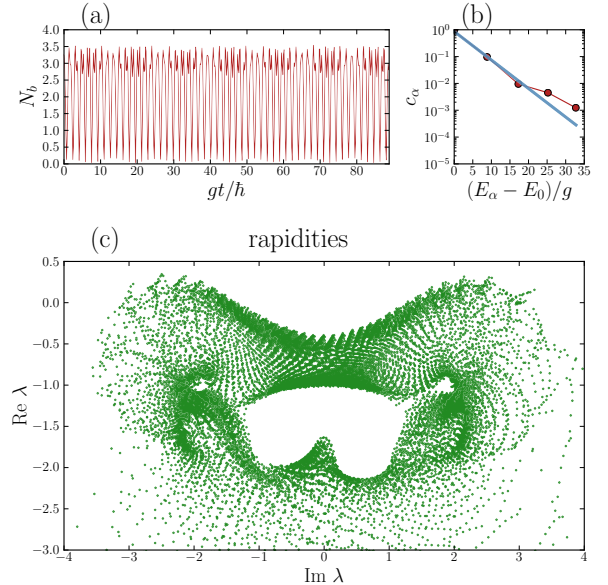


FIG. 2: The Dicke model driven near-resonantly with amplitude $\Delta_0/g = 5$ and frequency $\omega = 3.68g/\hbar$. For explanations see caption of Fig. 1.

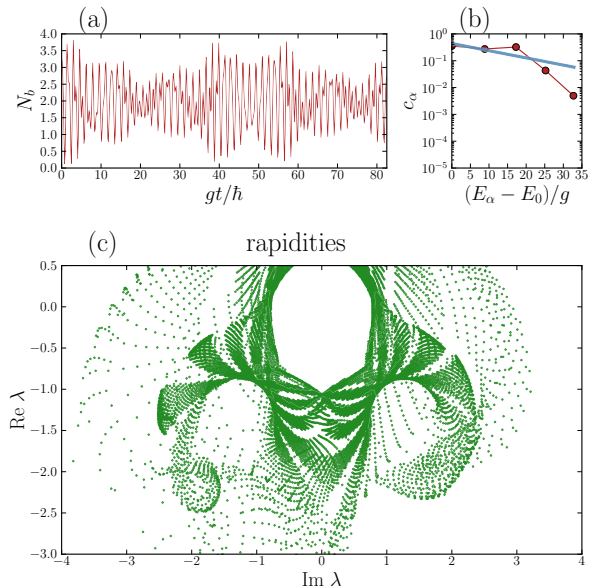


FIG. 3: The Dicke model driven with amplitude $\Delta_0/g = 5$ and frequency $\omega = 3.75g/\hbar$. For explanations see caption of Fig. 1.

In order to characterize the statistical properties of this system we measure the distribution of states averaged over all cycles

$$c_\alpha = \frac{1}{P} \sum_p |\langle \psi(t_p) | \alpha \rangle|^2, \quad (7)$$

where $|\psi(t_p)\rangle$ is a state of the driven system at $t = t_p$ and $|\alpha\rangle$ are the eigenstates of the Hamiltonian $H(t_p)$ after p cycles. In Fig. 1(b), we found their distribution to decay rapidly, which is expected in this nearly adiabatically driven system. We compare this distribution to a Boltzmann distribution, $c_\alpha = e^{-\beta E_\alpha}/Z$, with the same average energy. It turns out that the Boltzmann distribution cannot describe the weights (Fig. 1(b)).

In Fig. 2 we consider a slight increase of the frequency with respect to non-resonant case to $\omega = 3.68g/\hbar$. The boson occupation, which starts to exhibit an additional beating frequency (Fig 2(a)), suggests that a resonance is approached in the quantum model. Interestingly, this comes along with a scattering of the rapidities on the

collapsed 2-dimensional stroboscopic maps (Fig. 2(c)). From the point of view of the auxiliary classical system, these dynamics rather strongly deviate from the integrable limit. It has to be noted that despite the relatively dense exploration of the phase space, we could not find an indication of truly chaotic behavior. Nevertheless, and despite the small number of degrees of freedom in the system, this leads to a state distribution remarkably close to Boltzmann distribution (Fig 2(b)).

Further increasing the driving frequency to $\omega = 3.75g/\hbar$ as in Fig. 3, leads to strongly beating dynamics of boson occupancies. This resonance of the quantum model leads to a new structured pattern in the stroboscopic map of the classical variables. This hints that there are new quasi-periodic orbits emerging, which reside on a topological structure different from the one of the near-adiabatic case. The state distribution in Fig. 3(c) shows that there is a large amount of energy pumped into the system and the weights deviate considerably from the Boltzmann distribution.

Conclusions and outlook – In summary, we derived an correspondence between a time-dependent quantum model and an auxiliary classical system. The strength of this approach is illustrated by an example of a driven Dicke model with a frequency tuned from a non-resonant to a resonant value. The emerging dynamics can be interpreted in terms of the classical underlying system, whose trajectories lie on different KAM tori in non-resonant and resonant cases. At the point where one torus is deformed into the other, irregularity in the classical dynamics is most pronounced and time-averages of quantum observables approach thermal equilibrium.

One perspective of our method, which is not restricted to low-energy effective descriptions, would be an application to the high-energy part of quantum chromodynamics (QCD), which is described by the integrable quantum spin chain with complex spin [18]. The separation of variables for this model has been implemented in Ref. [19]. Dynamical perturbation from integrability may therefore shed a new light on phenomena of confinement. Moreover, integrable systems found in the context of AdS/CFT correspondence, can provide a basis to go beyond the limits where the duality is established.

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Supplementary Material

S1. SUMMARY

A single Bethe state is not sufficient to fully describe the problem of the time-dependent Schrödinger equation of a general Bethe ansatz solvable model. Here we demonstrate that a sufficient basis describing dynamics of Gaudin models can be given by using quantum version of separation of variables, also called functional Bethe ansatz (FBA). We use this technique to formulate a theory for dynamically perturbed Bethe ansatz solvable models. As a result, we get a Lagrangian of a classical system of equations of motion which governs the motion of rapidities. For sake of concreteness, we focus mostly on Gaudin-type models, but the approach we present here is rather general and we show how it may be extended to arbitrary integrable models.

In the first section of the Supplementary Information we summarize the construction the algebraic Bethe ansatz (ABA) and the FBA. On the basis of these tools we show a detailed derivation of our main results in the next sections. Then, we illustrate the principle of the method in a detailed discussion of the Dicke model.

S2. SEPARATION OF VARIABLES IN QUANTUM INTEGRABLE SYSTEM

A. Algebraic Bethe ansatz

We start with a general construction of the ABA and then focus on its reduction to Gaudin models [S1]. The ABA for the $sl(2)$ -related models (the class of models where the quantum space of individual particles is a representation space of the $sl(2)$ algebra) starts from the definition of the L -operator

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \quad (\text{S2})$$

where the elements are operators on the quantum space \mathcal{H} , which depend on a complex parameter λ , called a spectral parameter or rapidity. To ensure integrability, the $L(\lambda)$ -operator is imposed to satisfy the Yang-Baxter equation,

$$R_{12}(\lambda - \mu)L_1(\lambda)L_2(\mu) = L_2(\mu)L_1(\lambda)R_{12}(\lambda - \mu), \quad (\text{S3})$$

where the L -operators L_1 and L_2 act on copies of \mathcal{H} , \mathcal{H}_1 and \mathcal{H}_2 , and the quantum R -matrix acts on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Both, R -matrix and L -operators, can depend on several types of parameters: the spectral parameter λ , inhomogeneity parameters z_j (where j belongs to some countable set) and a parameter η , which controls the deviation from the quantum group structure. This implicit dependence on $\lambda, \{z_j\}$ and η is always assumed, even though only the spectral parameter λ is kept explicitly.

The generic R -matrix can be expanded as a regular series of the parameter η ,

$$R(\lambda) = I_{4 \times 4} + \eta r(\lambda) + O(\eta^2), \quad (\text{S4})$$

where the matrix $r(\lambda)$ is sometimes called quasiclassical r -matrix and $I_{4 \times 4}$ is the identity operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$. The dependence of r on $\{z_j\}$ is inherited from the corresponding R -matrix. In this limit, the Yang-Baxter equation takes the following form:

$$[L(\lambda) \otimes I, I \otimes L(\mu)] + [r(\lambda - \mu), L(\lambda) \otimes I + I \otimes L(\mu)] = 0. \quad (\text{S5})$$

In the case of Gaudin models one assumes the following form for the r -matrix:

$$r(\lambda) = \begin{pmatrix} f(\lambda) & 0 & 0 & 0 \\ 0 & 0 & g(\lambda) & 0 \\ 0 & g(\lambda) & 0 & 0 \\ 0 & 0 & 0 & f(\lambda) \end{pmatrix}. \quad (\text{S6})$$

Here we specialize on the case of rational functions $f(\lambda) = g(\lambda) = 1/\lambda$. This leads, in particular, to spin models with $SU(2)$ symmetry. Trigonometric or elliptic functions describe models with lower symmetry. Substituting the matrix form of the L -operator into the Yang-Baxter equation one can derive an algebra of A, B, C, D -operators [S2]. This algebra plays a fundamental role in the discussion of the method of separation of variables. To first order in η , and using the representation (S6), the expansion of the algebraic commutation relations (S5) lead to the Gaudin algebra,

$$\begin{aligned} [A(\lambda), B(\mu)] &= f(\lambda - \mu)B(\lambda) - g(\lambda - \mu)B(\mu), \\ [A(\lambda), C(\mu)] &= -f(\lambda - \mu)C(\lambda) + g(\lambda - \mu)C(\mu), \\ [B(\lambda), C(\mu)] &= 2(f(\lambda - \mu)A(\lambda) - g(\lambda - \mu)A(\mu)), \\ [B(\lambda), B(\mu)] &= 0, \\ [B(\lambda), C(\lambda)] &= A'(\lambda) - D'(\lambda), \end{aligned} \quad (\text{S7})$$

and $D(\lambda) = -A(\lambda)$. The generating function of the integrals of motion H_i up the first order in η is defined as

$$\begin{aligned} T(\lambda) &= \frac{1}{2} \text{Tr}[L^2(\lambda)] \\ &= \frac{1}{2} (A^2(\lambda) + D^2(\lambda) + B(\lambda)C(\lambda) + C(\lambda)B(\lambda)). \end{aligned} \quad (\text{S8})$$

For sake of concreteness we use a realization of operators $A(\lambda), B(\lambda), C(\lambda), D(\lambda)$ in terms of generators of the compact Lie algebra,

$$A(\lambda) = - \sum_{j=0}^N f(\lambda - z_j) S_j^z, \quad (\text{S9})$$

$$B(\lambda) = \sum_{j=0}^N g(\lambda - z_j) S_j^-, \quad (\text{S10})$$

$$C(\lambda) = \sum_{j=0}^N g(\lambda - z_j) S_j^+, \quad (\text{S11})$$

$$D(\lambda) = -A(\lambda). \quad (\text{S12})$$

We explicitly restore the dependence on the inhomogeneity parameters $\{z_j\}$ and ϵ_j . The total number of spins is $N + 1$, however, only N independent inhomogeneity parameters have to be considered; without restriction of generality one can set $z_0 = 0$. The operators S_j^a are the generators of the Lie algebra $su(2)$,

$$[S_j^1, S_k^2] = iS_j^3\delta_{jk}, [S_j^2, S_k^3] = iS_j^1\delta_{jk}, [S_j^3, S_k^1] = iS_j^2\delta_{jk}. \quad (\text{S13})$$

We denote $S_j^\pm = S_j^1 \pm iS_j^2$. Substituting Eqs. (S9) into Eq. (S8) we obtain for the rational case

$$T(\lambda) = \text{const} + \sum_j \frac{H_j}{\lambda - z_j} + \sum_j \frac{l_j(l_j + 1)}{(\lambda - z_j)^2}, \quad (\text{S14})$$

where $l_j(l_j + 1) = \mathbf{S}_j^2$ is the value of the Casimir operator of the representation on site j . The residues of $T(\lambda)$ represent Hamiltonians

$$H_j = \sum_{k \neq j} \frac{\mathbf{S}_j \cdot \mathbf{S}_k}{z_j - z_k}. \quad (\text{S15})$$

Particular examples are the central spin models with $j = 0$ and $1/z_j$ being the potential between spins, or Dicke models, describing interactions between bosons and spins, which can be obtained in the limit $l_0 \rightarrow \infty$.

Bethe states, i.e. eigenstates of conserved quantities, are obtained by applying the product of C -operators to the vacuum state $|0\rangle$ (defined by $B(\lambda)|0\rangle = 0$),

$$|\Psi(\lambda_1, \dots, \lambda_M)\rangle = \prod_{n=1}^M C(\lambda_n)|0\rangle.$$

For the rational $sl(2)$ -related Gaudin models the rapidities λ_m , $m = 1, \dots, M$ have to fulfill the *Bethe equations*

$$\Lambda(\lambda_m) = \sum_{n \neq m} \frac{1}{\lambda_m - \lambda_n}, \quad (\text{S16})$$

where the function $\Lambda(\lambda_m)$ depends on the set of inhomogeneities $\{z_j\}$ and can be represented as a ratio of two polynomials,

$$\Lambda(\lambda_m) = \sum_{n \neq m} \frac{a_n}{\lambda_m - z_j} + b + c\lambda_m + \dots \equiv \frac{P(\lambda_m)}{Q(\lambda_m)}, \quad (\text{S17})$$

where dots denote possible term of higher order in λ .

B. Separation of variables

We review the method of separation of variables to the classical and quantum systems. More details and further references can be found in the review paper of Sklyanin [S3] or in Ref. [S4].

Sklyanin [S3] suggested to introduce separated variables for the Bethe algebra as operators u_0 and u_j , $j = 1, \dots, N$, acting on the quantum space \mathcal{H} , such that

$$C(u_j) = 0, \quad \text{and} \quad C(\lambda) = u_0/\lambda. \quad (\text{S18})$$

Arguments of A, B, C and D are no more \mathbb{C} -numbers but operators. For general Bethe ansatz solvable systems this substitution can only be done by extending the algebra [S5], but for Gaudin-type models, the C matrix can be rewritten as

$$C(\lambda) \equiv \sum_{j=0}^N \frac{S_j^+}{\lambda - z_j} = \frac{u_0}{\lambda} \frac{\prod_{j=1}^N (\lambda - u_j)}{\prod_{j=1}^N (\lambda - z_j)} \quad (\text{S19})$$

and the separated variables can be given explicitly. It can be verified that u_0, u_j commute with each other. From the residue at $\lambda = z_j$ we find that the operator S_j^+ can be rewritten in terms of the basis of symmetric functions of separated variables,

$$S_j^+ = \text{res}_{\lambda \rightarrow z_j} C(\lambda) = u_0 \frac{\prod_{k=1}^N (z_j - u_k)}{\prod_{k \neq j}^N (z_j - z_k)}. \quad (\text{S20})$$

A set of conjugated variables is obtained by substituting variables u_j into diagonal elements of the L -matrix,

$$v_j = A(u_j). \quad (\text{S21})$$

For the algebra of Gaudin operators commutation relations become canonical:

$$[v_j, u_k] = \delta_{jk}. \quad (\text{S22})$$

Therefore, u_j and iv_j can be seen as pairs of position and momentum operators.

It is important to note that there is one conjugated pair less than spins in the system ($j = 1, \dots, N$). The operator u_0 is special and does not come together with a conjugated operator.

From the form of the transfer matrix $T(u) = \frac{1}{2} \text{Tr}(L^2(u))$ and the commutation relations one can define a realization of the v_j -operators in terms of the u_j variables as

$$v_j = \frac{\partial}{\partial u_j} - \Lambda(u_j), \quad (\text{S23})$$

where the function $\Lambda(u_j)$ forms a spectrum of operator u_j . They are given by the vacuum eigenvalues of the operator $A(\lambda)$, $A(\lambda)|0\rangle = \Lambda(\lambda)|0\rangle$ and is therefore specific for every model. We note, however, that the separation of variables does not need a well-defined pseudovacuum state, like in the algebraic Bethe ansatz, and the function $\Lambda(u)$ can be determined by operator techniques once we identified the space of states on which the operators u_j, v_j act [S3].

1. Coordinate representation

In order to express all the operators of the model in terms of the new variables u_j and v_j , one considers first a representation of the $sl(2)$ algebra in coordinates ξ_j $j = 0, \dots, N$ of some Euclidean space and corresponding differential operators $\partial/\partial\xi_j$:

$$S_j^3 = -\xi_j \frac{\partial}{\partial\xi_j} + l_j, \quad S_j^+ = \xi_j, \quad S_j^- = \xi_j \frac{\partial^2}{\partial\xi_j^2} - 2l_j \frac{\partial}{\partial\xi_j}. \quad (\text{S24})$$

The Gaudin Hamiltonian then reads

$$H_j = \sum_{k=0}^N \frac{\mathbf{S}_k \cdot \mathbf{S}_j}{z_k - z_j} \quad (\text{S25})$$

$$= \sum_{k=0}^N \frac{-\xi_j \xi_k (\partial_{\xi_k} - \partial_{\xi_j})^2 - (l_j \xi_k - l_k \xi_j) (\partial_{\xi_k} - \partial_{\xi_j}) + l_j l_k}{z_j - z_k}.$$

Since in this representation the spin operator S_j^+ is given by the coordinate, differentiating the log of both parts of the Eq. (S19) we obtain

$$\left(\frac{du_0}{u_0} + \sum_{j=1}^N \frac{du_j}{u_j - \lambda} \right) C(\lambda) = \sum_{j=0}^N \frac{d\xi_j}{\lambda - z_j}. \quad (\text{S26})$$

In particular, this implies that

$$\frac{d\xi_j}{\xi_j} = \frac{du_0}{u_0} + \sum_{k=1}^N \frac{du_k}{u_k - z_j}. \quad (\text{S27})$$

Using these equations one can establish the following identities:

$$\frac{\partial}{\partial u_j} = \sum_{k=0}^N \frac{\xi_k}{u_j - z_k} \frac{\partial}{\partial \xi_k}, \quad u_0 \frac{\partial}{\partial u_0} = \sum_{k=0}^N \xi_k \frac{\partial}{\partial \xi_k}, \quad (\text{S28})$$

and

$$\frac{\partial \xi_j}{\partial u_k} = \frac{\xi_j}{u_k - z_j}, \quad \frac{\partial \xi_j}{\partial u_0} = \frac{\xi_j}{u_0}. \quad (\text{S29})$$

These equations can be used to perform a change of variables between $\{\xi_j; \frac{\partial}{\partial \xi_j}\}$ and $\{u_0, u_j; \frac{\partial}{\partial u_0}, \frac{\partial}{\partial u_j}\}$ including the Jacobian. Thus, taking the pole at $\lambda = u_j$ and writing $du_j = \sum_{k=1}^N (\partial u_j / \partial \xi_k) d\xi_k$ one obtains from (S26)

$$\frac{\partial u_j}{\partial \xi_k} = -\frac{1}{u_0} \frac{\prod_{p=1}^N (u_j - z_p)}{\prod_{p \neq j}^N (u_j - u_p)}. \quad (\text{S30})$$

Using (S30) one finds

$$\partial_{\xi_i} = \partial_{u_0} - \frac{1}{u_0} \sum_{j=1}^N \frac{\prod_{k \neq i}^N (u_j - z_k)}{\prod_{k \neq j}^N (u_j - u_k)} \partial_{u_j}, \quad (\text{S31})$$

$$\partial_{\xi_i} - \partial_{\xi_j} = \frac{z_j - z_i}{u_0} \sum_{k=1}^N \frac{\prod_{p=1}^N (u_k - z_p)}{\prod_{p \neq k}^N (u_k - u_p)} \partial_{u_k}, \quad (\text{S32})$$

and, from Eq. (S28),

$$\partial_{u_j}^2 = \sum_{i=0}^N \sum_{k=0}^N \frac{\xi_j \xi_k}{(u_j - z_i)(z_i - z_k)} (\partial_{\xi_j} - \partial_{\xi_k})^2, \quad (\text{S33})$$

$$\sum_{i=1}^N \frac{l_i}{u_j - z_i} \partial_{u_i} = \sum_{i=0}^N \sum_{k=0}^N \frac{l_i \xi_k - l_k \xi_j}{(u_j - z_i)(z_i - z_k)} (\partial_{\xi_j} - \partial_{\xi_k}), \quad (\text{S34})$$

where the identity

$$\sum_{i=0}^N \sum_{j \neq i}^N \frac{1}{(u - z_i)(u - z_j)} = \left(\sum_{i,j=0}^N \frac{1}{(u - z_i)(u - z_j)} - \sum_{i=0}^N \frac{1}{(u - z_i)^2} \right) = 2 \sum_{i=0}^N \sum_{j \neq i}^N \frac{1}{z_i - z_j} \frac{1}{u - z_i}. \quad (\text{S35})$$

has been used. With these formulas one finds the representation of the spin operator

$$S_i^3 = \frac{\prod_{k=1}^N (z_i - u_k)}{\prod_{s \neq i}^N (z_i - z_s)} \left(u_0 \partial_{u_0} - \sum_{j=1}^N \frac{\prod_{k=0}^N (u_j - z_k)}{\prod_{k \neq j}^N (u_j - u_k)} v_j \right), \quad (\text{S36})$$

where we used the Eqs. (S30) and (S29) and where v_j is defined in Eq. (S23) with $\Lambda(u) = \sum_{p=0}^N \frac{l_p}{u - z_p}$. It is instructive to compare it with the result of deriving it from the pole of $A(\lambda)$. Indeed,

$$A(\lambda)|_{\lambda=u_j} = \left(\sum_{k=0}^N \frac{S_k^z}{\lambda - z_k} \right)_{\lambda=u_j} = \left(\sum_{k=0}^N \frac{\xi_k \frac{\partial}{\partial \xi_k} - l_k}{\lambda - z_k} \right)_{\lambda=u_j} = \sum_{k=0}^N \frac{\xi_k}{u_j - z_k} \frac{\partial}{\partial \xi_k} - \Lambda(u_j) = \frac{\partial}{\partial u_j} - \Lambda(u_j) \equiv v_j. \quad (\text{S37})$$

Using these formulas and Eq. (S14) together with Eq. (S8) we establish that the value of the generating function $T(\lambda)$ taken at $\lambda = u_j$ (remember that $C(u_j) = 0$) is

$$\begin{aligned} T(u_j) &= \sum_{i=0}^N \frac{H_i}{u_j - z_i} + \sum_{i=0}^N \frac{l_i(l_i + 1)}{(u_j - z_i)^2} \\ &= \frac{\partial^2}{\partial u_j^2} - 2\Lambda(u_j) \frac{\partial}{\partial u_j} + \Lambda^2(u_j) - \Lambda'(u_j) \\ &= \left(\frac{\partial}{\partial u_j} - \Lambda(u_j) \right)^2 = v_j^2. \end{aligned} \quad (\text{S38})$$

Since v_j can be interpreted as a canonical variable conjugated to u_j , the expression for $T(u_j)$ looks exactly as a Hamiltonian for the free particle propagating in the background of some monopole (vortex) distribution in the 2D plane.

The explicit expression for the commuting Hamiltonians in terms of separated variables can be obtained either directly, substituting the explicit change of variables through Eqs. (S31), (S32) into (S25),

$$H_i = \frac{\prod_{k=1}^N (z_i - u_k)}{\prod_{s \neq i}^N (z_i - z_s)} \sum_{i=1}^N \frac{\prod_{n=0}^N (u_i - z_n)}{\prod_{n \neq k}^N (u_i - u_n)} v_i^2. \quad (\text{S39})$$

Using the Lagrange interpolating formula, the gener-

ating function for the integrals of motion in terms of the separated variables at arbitrary λ takes the following form

$$T(\lambda) = \sum_{k=1}^N \frac{Q(\lambda) a(u_k)}{Q'(u_k) (\lambda - u_k) a(\lambda)} v_k^2, \quad (\text{S40})$$

where

$$\begin{aligned} Q(\lambda) &= \prod_{i=1}^N (\lambda - u_i), \quad Q'(u_k) = \prod_{\substack{p=1 \\ p \neq k}}^N (u_k - u_p), \\ a(\lambda) &= \prod_{k=0}^N (\lambda - z_k). \end{aligned} \quad (\text{S41})$$

One can now define a generating function for the eigenvalues of the generating function via

$$\tau(\lambda) = \sum_{i=0}^N \frac{h_i}{\lambda - z_i} = \frac{e(\lambda)}{a(\lambda)}, \quad (\text{S42})$$

where $a(\lambda)$ is defined in Eq. (S41). Therefore, the stationary Schrödinger equation with $T(\lambda)$ defined in (S38) in separated variables takes the form of a one-dimensional Schrödinger equation

$$a(u_j) \left(\frac{\partial^2}{\partial u_j^2} - 2\Lambda(u_j) \frac{\partial}{\partial u_j} + \Lambda^2(u_j) - \Lambda'(u_j) \right) \Psi(u_0, \dots, u_N) = e(u_j) \Psi(u_0, \dots, u_N), \quad (\text{S43})$$

what proves that the wavefunction can be factorized,

$$\Psi(u_0, \dots, u_N) = \prod_{j=0}^N \psi(u_j) \quad (\text{S44})$$

where the operator u_0 is now included into the separated variables. Moreover the same form of a stationary Schrödinger equation for the generating function $T(\lambda)\Psi(\lambda) = \tau(\lambda)\Psi(\lambda)$ is obtained from (S40) by taking the residues at poles at both sides. We note that the form of this equation is the same for every u_j , $j = 0 \dots N$.

Several remarks about the method of separation of variables are at order: (i) The separated equation for variable u_j depends only on the variable u_j . (ii). The separated equations have the same form for all u_j . (iii) The differential operator has the form of Baxter Q -operator equation. (iv) Physically, separated variables can be thought as the eigenmodes of the spin densities. Indeed, we can write

$$\frac{\partial}{\partial \lambda} \sum_{j=1}^N \frac{S_j^a}{\lambda - z_j} = \sum_{j=0}^N S_j^a \delta(\lambda - z_j). \quad (\text{S45})$$

In other words, separated variables serve as a set of normal coordinates in the space of operators.

Now we consider a time-dependent Schrödinger equation,

$$\frac{\partial}{\partial t} \Psi(\{\lambda\}, t) = T(\lambda) \Psi(\{\lambda\}, t) \quad (\text{S46})$$

and make again a product ansatz,

$$\Psi(\{\lambda\}) = \prod_{j=0}^N \psi(u_j). \quad (\text{S47})$$

Substituting a time-depending ansatz we obtain a separated equation

$$i\partial_t \psi(u_j) = a(u_j) \left(\frac{\partial^2}{\partial u_j^2} - 2\Lambda(u_j) \frac{\partial}{\partial u_j} + \Lambda^2(u_j) - \Lambda'(u_j) \right) \psi(u_j). \quad (\text{S48})$$

Now, since the equations are the same for all u_k we denote the variable u_k by u and consider the following change of variables

$$x = f(u) = \int^u \frac{dy}{\sqrt{a(y)}}, \quad (\text{S49})$$

where the integration should be understood in general as a contour integral which avoids the singularities of $a(y)$. Next we implement this transformation to the wave function (change of variables and multiplication by the factor). The Hamiltonian can be now put into the form of a one-dimensional Schrödinger equation with a Hamiltonian

$$H(x) = \partial_x^2 + V(x). \quad (\text{S50})$$

Instead of doing this in general we will postpone this to explicit examples below. We only remark that the form of the potential is rather special and allows for supersymmetrization, meaning that the function $\Lambda(u)$ can be considered as a derivative of the superpotential $W(u)$, and the Hamiltonian in Eq (S48) is the bosonic part of the Hamiltonian of supersymmetric quantum mechanics.

To take a time-dependence into account we will follow a procedure of moving zeroes as described in the book of Calogero [S6]. In this approach a wave function is assumed to have the following product form,

$$\psi(u, t) = \phi(t) \prod_{n=1}^M (u - \lambda_n(t)), \quad (\text{S51})$$

where $\lambda_n(t)$ is a pole of the wavefunction. For this product form one obtains

$$\begin{aligned} \frac{\partial}{\partial u} \psi(u, t) &= \psi(u, t) \sum_{n=1}^M (u - \lambda_n(t))^{-1}, \\ \frac{\partial}{\partial t} \psi(u, t) &= \psi(u, t) \sum_{n=1}^M (u - \lambda_n(t))^{-1} (-\dot{\lambda}_n(t)), \\ \frac{\partial^2}{\partial u^2} \psi(u, t) &= 2\psi(u, t) \sum_{n=1}^M (u - \lambda_n(t))^{-1} \\ &\quad \left[\sum_{m=1, m \neq n}^M (\lambda_n(t) - \lambda_m(t))^{-1} \right]. \end{aligned} \quad (\text{S52})$$

Substituting this into Eq. (S48) and equating the residues at the poles we obtains

$$-i\dot{\lambda}_n(t) = 2a(\lambda_n) \sum_{m \neq n} \frac{1}{\lambda_n - \lambda_m} - 2a(\lambda_n)\Lambda(\lambda_n). \quad (\text{S53})$$

At equilibrium, $\dot{\lambda}_n = 0$, we recover the Bethe equation in the form of Eq. (S16).

Now we are going to derive that equations (S53) can be considered as equations of motion for certain Lagrangean dynamical system. Indeed if we denote the right hand side of Eq. (S53) by $\Phi_n(\{\lambda\})$ and differentiate the equation $-i\dot{\lambda}_n(t) = \Phi_n(\{\lambda\})$ we obtain

$$\ddot{\lambda}_n = i \sum_{m=1}^M \frac{\partial \Phi_n}{\partial \lambda_m} \dot{\lambda}_m = - \sum_{m=1}^M \Phi_m \frac{\partial \Phi_n}{\partial \lambda_m}. \quad (\text{S54})$$

On the other hand if we consider the Lagrangean for M particles

$$\mathcal{L} = \sum_{n=1}^M (\dot{\lambda}_n)^2 - \sum_{n=1}^M (\Phi_n(\{\lambda\}))^2, \quad (\text{S55})$$

regarding the set $\{\lambda_n(t)\}$ as generalized coordinates, one can derive the equations of motion,

$$\ddot{\lambda}_n = \frac{1}{2} \frac{\partial \mathcal{L}}{\partial \lambda_n} = - \sum_{m=1}^M \Phi_m \frac{\partial \Phi_m}{\partial \lambda_n}. \quad (\text{S56})$$

We note that equations, (S54) and (S56) hold only if

$$\frac{\partial \Phi_n}{\partial \lambda_m} - \frac{\Phi_m}{\partial \lambda_n} = 0. \quad (\text{S57})$$

d is an exterior derivative. For the gaudin models, one can show that $\Phi_n = \partial_n \phi(\{\lambda_k\})$ with some function $\phi(\{\lambda_k\})$, therefore the 1-form is always closed. If one makes the change of variables (S49) from variable λ_n to variables x_n the equation of motions acquire the form $-i\dot{x}_n = \partial_{x_n} \phi$. Such, one establishes a M -particle Lagrangean system whose Hamiltonian is given by

$$H = \sum_{n=1}^M p_n^2 + V_M(\{x_n\}). \quad (\text{S58})$$

where $p_n = \dot{x}_n$. Substituting directly the function $\Phi(\{\lambda\})$ into the Lagrangean gives for the potential, in terms of $\{\lambda_n\}$ (explicitly obtained by expanding $\sum_n (\Phi_n^2/a_n)$),

$$V_M(\{\lambda\}) = 4 \sum_{n=1}^M a_n \Lambda_n^2 + 8 \sum_{n,m=1; n \neq m}^M \frac{a_n \Lambda_n}{\lambda_n - \lambda_m} - 4 \sum_{n=1}^M a_n \left(\sum_{m \neq n} \frac{1}{\lambda_n - \lambda_m} \right)^2, \quad (\text{S59})$$

where $a_n \equiv a(\lambda_n)$ and $\Lambda_n \equiv \Lambda(\lambda_n)$. Note that after expressing this in terms of λ_n 's we have to make a change of variables to $x = \int a^{-1/2}(\lambda) d\lambda$ and express V_M in terms of them. For generic functions $a_n(\lambda), \Lambda_n(\lambda)$ we may get rather complicated potential. We will show an explicit form of the potential V_M for the Dicke model below which gives an integrable classical model.

S3. SOLUTION OF THE DICKE MODEL AT EQUILIBRIUM AND WITH TIME-DEPENDENT DETUNING

A. Equilibrium BA

In this section we are going to discuss the details of the mapping dynamics of the quantum Dicke Hamiltonian

$$H_D = \Delta S^z + g(b^\dagger S^- + b S^+), \quad (\text{S60})$$

to the classical equations of motion. The total spin is $\sum_{a=1}^3 (S^a)^2 = S(S+1)$ and the conserved quantity is $M = b^\dagger b + S^z + S$. It can be shown [S7], that this model belongs to the Gaudin algebra with

$$C(\lambda) = b^\dagger - \frac{S^\dagger}{\lambda}, \quad (\text{S61})$$

$$|0\rangle = |0\rangle_b |S, -S\rangle. \quad (\text{S62})$$

At equilibrium the Bethe equations read

$$\frac{2S}{\lambda_n} - \lambda_n + \Delta - \sum_{\substack{m=1 \\ m \neq n}}^M \frac{2}{\lambda_n - \lambda_m} = 0, \quad (\text{S63})$$

and define the eigenenergies

$$E_{S,M} = -S(\Delta + 2 \sum_{n=1}^M \frac{1}{\lambda_n}) = \Delta(M - S) - \sum_{n=1}^M \lambda_n. \quad (\text{S64})$$

B. Time-dependent perturbation

A natural way to drive the Dicke model out of equilibrium is to apply some time-dependent perturbation.

For time-dependent detuning we obtain a solution of the Schrödinger equation from a Bethe wavefunction in which static spectral parameters are replaced by time-dependent ones and then derive the underlying classical system of equations of motion. We make the ansatz

$$|\Psi(t)\rangle = \exp[ie(t)] \prod_{n=1}^M C(\lambda_n(t)) |0\rangle, \quad (\text{S65})$$

where

$$\begin{aligned} e(t) &= \int_0^t \text{res}_{\mu \rightarrow 0} \Theta(\mu, \tau) d\tau, \\ \Theta(\mu, \tau) &= x^2(\mu) - \frac{\partial x(\mu)}{\partial \mu} - 2 \sum_{n=1}^M \frac{x(\mu) - x(\lambda_n(\tau))}{\mu - \lambda_n(\tau)}, \\ x(\mu) &= \frac{1}{2} \left(\frac{2S}{\mu} - \mu + \Delta \right), \\ e(t) &= S \int_0^t (\Delta + 2 \sum_{n=1}^M [\lambda_n(\tau)]^{-1}) d\tau. \end{aligned} \quad (\text{S66})$$

$e(t)$ is the time-dependent energy. The function $x(\mu)$ is defined by the vacuum state. The spectral parameters are now subject to the following set of equations

$$i \frac{\dot{\lambda}_n(t)}{\lambda_n(t)} = 2 \left[x(\lambda_n(t)) - \sum_{\substack{m=1 \\ m \neq n}}^M \frac{1}{\lambda_n(t) - \lambda_m(t)} \right]. \quad (\text{S67})$$

In what follows we show that the corresponding classical auxiliary system is the Inosemtev model. After the change of variables (S49)

$$\lambda_n = x_n^2, \quad x_n \rightarrow x_n/2, \quad (\text{S68})$$

the equation for the dynamical BA equation takes the following form:

$$-i\dot{x}_n = ax_n^3 + bx_n + \frac{c}{x_n} + 2x_n \sum_{m \neq n} \frac{1}{x_n^2 - x_m^2}. \quad (\text{S69})$$

The constants are $a = 1/8$, $b = -\Delta/2$, $c = -4S$. Considering now the equation

$$\begin{aligned} \ddot{x}_n &= \frac{c^2}{x_n^3} - (2ac + b^2 + 2a(M-1))x_n - 4abx_n^3 - 3a^2x_n^5 \\ &+ \frac{1}{2} \sum_{\substack{m=1 \\ m \neq n}}^M \left(\frac{1}{(x_n - x_m)^3} + \frac{1}{(x_n + x_m)^3} \right), \end{aligned} \quad (\text{S70})$$

one recovers the potential of the classical model:

$$\begin{aligned} H_I &= \sum_{n=1}^M \frac{p_n^2}{2} + \frac{1}{2} V(\{x_n\}), \\ V(\{x_n\}) &= \sum_{\substack{m=1 \\ m \neq n}}^M \left(\frac{1}{(x_n - x_m)^2} + \frac{1}{(x_n + x_m)^2} \right) \\ &+ \frac{1}{64} x_n^6 + \beta x_n^4 + \gamma x_n^2 + \sum_{n=1}^M \frac{16S^2}{x_n^2}. \end{aligned} \quad (\text{S71})$$

The parameters are given by

$$\beta = -\frac{\Delta(t)}{8}, \quad \gamma = \frac{1}{4}(\Delta^2(t) - \dot{\Delta}(t) - 4S + M - 1). \quad (\text{S72})$$

This model is known as BC-type Inozemtsev model [S8]. It is integrable on the classical level for time-independent parameters. We note that this is a *complexified* version of the *classically* integrable Inozemtsev (or BC-type) model.

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